Announcement

1) Scholarships app due today

2) HW due Thursday

Notation: If the limit of f at $a = x$ is equal to L, we write

$$\boxed{\lim_{x \to a} f(x) = L}$$

Proposition: (properties)

Suppose f and g are defined on $\overline{S}$, $S \subseteq \mathbb{R}$, then if $a \in S'$ and $c \in \mathbb{R}$, then

if $\lim_{x \to a} f(x) = L$ and

$\lim_{x \to a} g(x) = M$

1) $\lim_{x \to a} (f(x) \pm g(x)) = L \pm M$

2) $\lim_{x \to a} (c f(x)) = cL$

3) $\lim_{x \to a} (f(x) \cdot g(x)) = LM$

4) $$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M} \text{ if } M \neq 0$$

proof: all immediate by using the sequential formulation for limits and the corresponding properties 1) - 4) for sequences since we already proved it then!

$\square$

Section 4.3

Definition: (continuity, metric space)

Let (X, d_1) and (Y, d_2) be two metric spaces.

$f: X \to Y$ is continuous at $x \in X$ if for every $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$\boxed{d_2(f(x), f(y)) < \varepsilon \text{ whenever } d_1(x, y) < \delta}$$

Comments

1) if $X = Y = \mathbb{R}$ with $d_1 = d_2 =$ usual metric, continuity becomes: $\forall\, \varepsilon > 0, \exists\, \delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ when $|x - y| < \delta$.

2) $x = y$ is allowed in the definition: continuity sees what happens at the point in question.

3) If f is continuous $\forall\, x \in X$, we say f is continuous.

Example 1: (a continuous function)

Let $X = Y = \mathbb{R}$ with usual metric

Define, for $x \geq 0$,

$$f(x) = \sqrt{x}.$$

Show f is continuous on its domain.

Case 1. $a = 0$

We only look at $x \geq 0$, intervals will be half-open.

Want: $\forall\, \varepsilon > 0,\ \exists\, \delta > 0$ such that

$$|\sqrt{x} - \sqrt{0}| < \varepsilon \text{ when}$$

$$|x - 0| < \delta.$$

This becomes

$$\sqrt{x} < \varepsilon \text{ when}$$

$$0 \leq x < \delta.$$

Start with "$\sqrt{x} < \varepsilon$".

try to manipulate this expression until you see x (without square root)

$$\sqrt{x} < \varepsilon$$

Multiply both sides by $\sqrt{x}$

$$x < \varepsilon \sqrt{x}$$

Divide by $\sqrt{x}$ (formally)

to get $\boxed{\dfrac{x}{\sqrt{x}} < \varepsilon}$ Show this, by choosing δ.

$\frac{x}{\sqrt{x}}$, choose δ so that

$x < \delta$ implies

$\frac{x}{\sqrt{x}} < \varepsilon$.

Choose $\boxed{\delta = \varepsilon^2}$.

Then if $x < \delta = \varepsilon^2$,

taking square roots,

$\boxed{\sqrt{x} < \varepsilon}$.

Case 2: $x > 0$

Want to show: $\forall \varepsilon > 0$, there exists $\delta > 0$ with

$$|\sqrt{x} - \sqrt{y}| < \varepsilon \text{ when}$$

$$|x - y| < \delta.$$

Start with $|\sqrt{x} - \sqrt{y}| < \varepsilon$.

Manipulate until you see

$$|x - y|.$$

$$|\sqrt{x}-\sqrt{y}|$$

$$= \left| (\sqrt{x}-\sqrt{y}) \cdot \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}} \right|$$

$$= \frac{|x-y|}{\sqrt{x}+\sqrt{y}}$$

$$\le \frac{|x-y|}{\sqrt{x}}$$ since $\sqrt{x}+\sqrt{y} \ge \sqrt{x}$

$< \varepsilon$?

Let $\delta = \varepsilon\sqrt{x}$ since we're fixing x, this is OK, but not δ depends on x!

Then

$$|\sqrt{x}-\sqrt{y}| \leq \frac{|x-y|}{\sqrt{x}}$$

$$< \frac{\cancel{\sqrt{x}} \cdot \varepsilon}{\cancel{\sqrt{x}}} = \varepsilon .$$

This shows $f(x)=\sqrt{x}$ is continuous for all $x \geq 0$.

This is almost correct! Need to correct for when ε is large by setting

$$\delta = \min\{\varepsilon\sqrt{x}, \sqrt{x}\}$$

Example 2: (bad example)

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

is nowhere-continuous since the limit does not exist at any $x \in \mathbb{R}$!

Theorem: Let X and Y be metric spaces. Then $f: X \to Y$ is continuous at $x \in X$ if and only if for every open set $O \subseteq Y$ with $f(x) \in O$, $f^{-1}(O)$ is an open set.

Proposition: (sums, products, etc)

Let $f: \mathbb{R} \to \mathbb{R}$ be continuous at $x \in \mathbb{R}$ (usual metric). Then if g is also continuous at x, we get

1) $f \pm g$ is continuous at x

2) $f \cdot g$ is continuous at x

3) cf is continuous at x $\forall c \in \mathbb{R}$

4) $\frac{f}{g}$ is continuous at x if $g(x) \neq 0$.

From proposition, any polynomial is continuous

polynomial:

$$f(x) = \sum_{k=0}^{n} a_k x^k$$

with $a_0, a_1, \ldots, a_n \in \mathbb{R}$.